Tuning and Design of Single-Input, Single-Output Control Systems for Parametric Uncertainty

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Frequency domain methods, which are for the design and tuning of control systems for processes with uncertain parameters, and are easy to use and interpret, are proposed. The advantages of the proposed methods over other H_∞ methods are mainly that they require substantially less expertise on the part of the designer, and that they result in control system performance that closely approximates that which was desired. Several examples illustrate the proposed methods. A stable second-order lag and dead time process, with gain and dead time uncertainties, provides the vehicle for studying controller designs for single and two degree of freedom control systems. Application of Mp tuning to an uncertain, unstable, first-order lag, plus dead time process, shows that there can be a significant performance penalty associated with using a simple feedback control system rather than an internally stable model-based control system. The performance of a controller for an integrating, oscillatory process obtained by Mp synthesis also compares favorably with that of a controller obtained by other investigators using loop shaping..

Introduction

Good process control requires that the control system give acceptable responses to setpoint changes and disturbances over the entire range of process operating conditions. Consequently, good control-system design and tuning methods must allow for the fact that the local process model, and hence the local control-system behavior, changes as operating conditions change. For guaranteed good control-system performance, it is generally not sufficient to design or tune a control system for a single, nominal model. However, if process operating conditions change gradually, then it is often sufficient to design and/or tune a control system so that it performs satisfactorily for the set of all linear constant-coefficient models that describe process operation at all operating conditions. Such sets are said to describe an uncertain process.

A great deal of effort has been expended, mainly in the last 20 years, to ensure satisfactory control-system performance for uncertain processes described by sets of linear, constant-coefficient models in the form of differential, differential−difference, and partial differential equations. Much of this effort has been motivated by Kharitonov's elegant theorem on the roots of sets of polynomials whose coefficients lie in fixed ranges (Kharitonov, 1978; Barmish, 1988; Bartlett, 1987; Bhattacharyya, 1988; Sideris and Gaston, 1986; Chapellat et al., 1995), and aims at determining the stability of the control system. Since stability alone does not ensure acceptable performance, however, other investigators (Doyle and Stein, 1981; Laughlin et al., 1986; Morari and Zafiriou, 1989; Braatz et al., 1996) have concentrated on assuring that the control system satisfies *a priori* frequency-domain specifications in anticipation that this will also ensure satisfactory time-domain performance.

In spite of the development of extensive software to assist in the design of control systems for uncertain processes via H_∞ methods (such as Matlab's Robust Control, and μ-Synthesis toolboxes), all such methods require substantial expertise on the part of the control system designer to obtain good control system designs. The designer must first decide which closed-loop transfer function (such as the sensitivity or

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complementary sensitivity function) is best suited for the design. The designer must then supply the software with frequency-dependent *a priori* bounds on the desired closed-loop performance. However, since closed-loop performance is strongly influenced by the level of uncertainty, it is at best an iterative process to provide a feasible set of bounds. Further, loop shaping and μ-Synthesis methods usually result in high-order controllers that must be then simplified, and while there is software to assist in the simplification, the designer has to be familiar with that software. The designer must also modify the process description so that the uncertainty description stands alone in a feedback loop. Again, there is software to carry out the needed linear fractional transformations, but these operations also require a background that is often lacking.

Another problem in designing and tuning control systems for uncertain processes is that of selecting the appropriate closed-loop transfer function for controller design. Most of the H_∞ designs found in the literature use the sensitivity function (that is, the transfer function between the output and the disturbance) for controller design. This practice is most likely due to the fact that placing a frequency-dependent upper bound on the sensitivity function also places a lower bound on the slowest time-domain responses. The problem here, however, is that the trade-off between speed of response and overshoot and/or oscillatory behavior is not easy to quantify based on the frequency response of the sensitivity functions. For example, the same upper bound on the sensitivity function leads to different maximum overshoots to a step setpoint, changes for different processes and models. Further, a particular frequency-dependent upper bound on the sensitivity function is not necessarily achievable by any control system. Again, the designer needs significant experience in order to provide achievable specifications.

For single-degree-of-freedom control systems, we propose placing a constant upper bound on the maximum magnitude of the transfer function between the process output and the setpoint (that is, on the complementary sensitivity function) over all frequencies and all uncertain parameters. It is relatively easy to relate the maximum magnitude of the frequency reponse of the complementary sensitivity function to the maximum time-domain overshoot to a step setpoint change. The controller design problem then becomes finding the controller that moves the *lower bound* of the complementary sensitivity function over all frequencies and all uncertain parameters as far to the right as possible, subject to a specified maximum magnitude on the upper bound. That is, we attempt to find the controller that yields the fastest possible responses for the slowest closed-loop responses, all the while ensuring that no response has more than the allowable overshoot to a step setpoint change. For two-degrees-of-freedom control systems for stable processes we propose using a modified version of the sensitivity function (which we call the partial sensitivity function) to design and tune the control systems for good disturbance rejection. The modified version of the sensitivity function has properties similar to those of the complementary sensitivity function. Consequently, the same tuning and design criteria can be used. For good setpoint responses, we again seek to achieve a specified upper bound on the maximum magnitude of the complementary sensitivity function. For unstable processes we propose that the appropriate design criteria for disturbance rejection and loop stabilization is to minimize the maximum magnitude of the sensitivity function. Once the controller for the feedback loop is in hand, a specified constant upper bound on the maximum magnitude of the complementary sensitivity function is used to tune the setpoint controller.

Achieving the closed-loop specifications discussed earlier requires maximizing or minimizing the appropriate closed-loop transfer function over all uncertain parameters and all positive frequencies. Of course, optimizations are actually carried out only over a quantized range of frequencies. An advantage of this procedure is that there is no need to approximate transcendental functions (such as exponentials that arise from dead times) with polynomials, as is commonly necessary using other H_∞ methods. However, since we propose to carry out designs that simply guarantee a finite upper bound on the magnitude of the frequency response of rather general transfer functions, it is necessary to provide the conditions under which such a finite upper bound guarantees that the closed-loop system is stable. We provide such conditions in the form of a Robust Stability theorem (Brosilow and Leitman, 2000). The proof of this theorem can be found in the foregoing reference. In this article we simply state the theorem, the conditions necessary for it to hold, and give a brief plausibility argument that we hope yields some insight as to why the theorem holds.

This article focuses on the tuning and synthesis of simple single-input, single-output IMC control systems. For such systems it is a relatively simple matter to convert the IMC control system to a simple feedback-control system. Further, it is often possible to approximate the feedback controller as a PID controller (Lee et al., 1998). It is also possible to extend the proposed approach to the tuning of multivariable systems [Stryczek and Brosilow (1996)].

The next section presents the Robust Stability theorem. The following sections present the controller design methods for one- and two-degrees-of-freedom control systems, along with several examples.

Robust Stability

The controller tuning and design methods of the following sections all rely on the fact that a closed-loop transfer function, $H(s, \alpha)$, given by

$$H(s,\alpha) \equiv \frac{g(s,\alpha)}{1+h(s,\alpha)} \tag{1}$$

is stable for all parameter vectors, α, contained in the parameter set Π if $H(s,\alpha)$ is stable for at least one parameter in the set Π, and if $\max_{\alpha \in \Pi} |H(i\omega,\alpha)|$ is finite $\forall \omega \geq 0$ and, the parameter set Π, and the transfer functions $g(s,\alpha)$ and $h(s,\alpha)$ satisfy the following conditions:

1. The parameter set Π is an *open, connected* subset of R^n (C^n denotes complex Euclidean n-space, and R^n denotes its real Euclidean n-subspace). Since Π is *connected*, for any two points $\tilde{\alpha}$ and $\tilde{\beta}$ in Π, there is a *path* σ *in* Π from $\tilde{\alpha}$ *to* $\tilde{\beta}$; that is, there is a continuous function σ: $[0, 1] \to \Pi$ such that $\sigma(0) = \tilde{\alpha}$ and $\sigma(1) = \tilde{\beta}$. Since Π is *open*, a path σ from $\tilde{\alpha}$ to $\tilde{\beta}$ can be taken arbitrarily smooth.

2. There is a *connected, open* set $\hat{\Pi}$ in C^n such that $\hat{\Pi} \cap R^n = \Pi$ and the functions $g(s, \alpha)$, *and* $h(s, \alpha)$ have meromorphic extensions to $C^+ \times \hat{\Pi}$ (if $h(s, \alpha)$ is "meromorphic for real parameters α" in the sense of convergent power-series expansions, this extension is always possible), also denoted by g and h, where R^n and C^n are real and complex n space and C^+ is the open complex half plane in C.

3. For each parameter $\alpha \in \Pi$, the function $h(s, \alpha)$ is not constant on C^+ (if $h(s, \alpha)$ is the Laplace transform of some real-valued *integrable* function and it is constant at α, it must be identically zero at α).

4. For each parameter $\alpha \in \Pi$, the function $h(s, \alpha)$ is real-valued or ∞ on the positive real axis $\{s: Re(s) > 0,\ Im(s) = 0\}$.

5. The functions $g(s, \alpha)$ and $h(s, \alpha)$ are (jointly) continuous functions from $\bar{P}^+ \times \Pi$ into P, where P is the complex projection sphere (Riemann sphere), and $\bar{P}^+$ represents the closure in P of the open hemisphere, P^+.

Conditions (2), (3), and (4) are regularity or consistency conditions on g and h. However, conditions (1) and (5) on the parameter set and the continuity of g and h, especially on the imaginary axis, are structurally necessary for robust stability to hold.

To gain some insight into why the robust stability result holds, let us assume that it is false. Then there is a parameter vector β_0, for which the system is stable, and another parameter vector α_0, for which the system is unstable. Since the parameter set Π is assumed open and connected, we can choose a path from α_0 to β_0 lying entirely inside the parameter set Π. Since the system is unstable for α_0, there is at least one point, say s_0, in $\overline{P^+}$ such that $1 + h(s_0, \alpha_0) = 0$. Since the system is stable for β_0, there is no such point corresponding to β_0. Then, if conditions (1) to (5) hold, we can find a parameter $\hat{\alpha} \in \Pi$ and a frequency $\hat{\omega} \in [0, \infty]$, such that $1 + h(i\hat{\omega}, \alpha) = 0$. Hence, $H(\hat{\omega}) = +\infty$, and the maximum magnitude of the closed-loop frequency response over all parameters is not bounded. This contradicts our hypothesis.

All the following sections use the preceding robust stability result. However, as we demonstrate, correct interpretation of the results on inherently unstable processes requires careful application of the theorem.

Design and Tuning of Single-Degree-of-Freedom Controllers

Mp tuning

Mp tuning for single-degree-of-freedom controllers aims to select the parameters of the controller, $c(s)$, in Figure 1, or

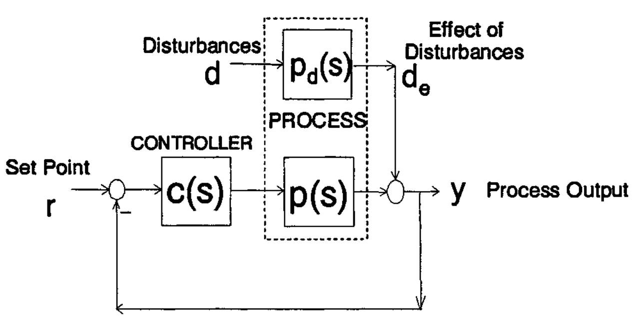

Figure 1. Single-degree-of-freedom feedback control system.

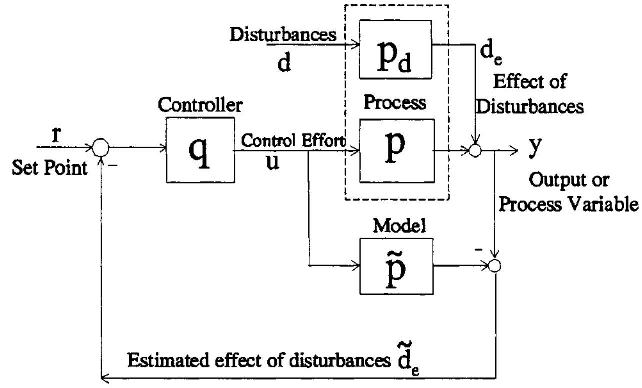

Figure 2. Internal model control system.

$q(s)$ and $\tilde{p}(s)$ in Figure 2 so that we have the following criteria.

Criterion 1. The magnitude of the complementary sensitivity function, $|\eta(i\omega)|$ $[\eta(s) \equiv y(s)/r(s)$ in Figures 1 and 2], as a function of frequency, ω, is equal to or less than a specified value, or Mp, for all processes, $p(s)$, in a predefined set Π. Also, for at least one process in Π the magnitude of $\eta(i\omega)$ must equal the specified Mp at one or more frequencies.

Criterion 2. No peak in $|\eta(i\omega)|$ vs. ω has more than a specified height above its nearest adjoining valley.

Criterion 1 can be expressed as:

$$|\eta(i\omega)| \le \text{Mp for all } p(s) \text{ in } \Pi \text{ and all } \omega \ge 0$$

and

$$|\eta(i\omega_c)| = \text{Mp for some } p(s) \text{ in } \Pi \text{ and some frequencies}, \ \omega_c. \tag{2}$$

Generally, the specified Mp is chosen based on the maximum desired overshoot, in the time domain, to a step setpoint change. For example, a Mp of 1.05 will yield approximately a 10% overshoot if the closed-loop system response can be reasonably approximated by that of a second-order system.

Criterion 2 attempts to maintain a desired relative stability for those closed-loop system responses that cannot be reasonably approximated by that of a second-order system. We have found through numerous examples that a maximum peak height of 0.1 usually prevents undesirable high-frequency oscillations in the process response to setpoint changes and disturbances.

Since the controller, $c(s)$, in Figure 1 generally has several adjustable parameters (such as three for a PID controller), there is generally no unique solution to the preceding Mp tuning problem. However, an IMC controller, $q(s)$, usually has only one tuning parameter, the IMC filter time constant. Therefore, our approach to controller design and tuning is to first design and tune an IMC controller, and then convert the IMC controller into a simple feedback controller using the techniques described by Lee et al. (1998).

Tuning IMC control systems

We select the filter time constant, ϵ, for the IMC controller, $q(s)$, of Figure 2 to satisfy Mp tuning Criteria 1 and 2. The complementary sensitivity function, η, for the IMC configuration of Figure 2 is

$$\eta(s) = y(s)/r(s) = \frac{q(s,\epsilon)p(s)}{1+(p(s)-\tilde{p}(s))q(s,\epsilon)}, \quad (3)$$

where

$q(s,\epsilon)$ = IMC controller transfer function, which depends on a filter time constant, ϵ

$\tilde{p}(s)$ = a "nominal" model in Π

Π = set of allowable processes.

The standard method for selecting the IMC controller, $q(s, \epsilon)$, is to select it to achieve a desired closed-loop response for the nominal process model, $\tilde{p}(s)$ (Morari and Zafiriou, (1989). This usually involves inverting the invertible part of the model, and reflecting any model right half-plane zeros about the imaginary axis in the Laplace domain. For any design, however, the IMC controller, $q(s, \epsilon)$, can always be written as

$$q(s,\epsilon) = \tilde{q}(s)f(\epsilon,s), \quad (4)$$

where

$f(\epsilon,s) = 1/(\epsilon s+1)^r$

r = order of $f(\epsilon,s)$

$\geq$ relative order of $\tilde{p}(s)$, usually r = relative order of $\tilde{p}(s)$

$\tilde{q}(s)$ = inverse of a portion of $\tilde{p}(s)$

$\tilde{q}(0) = \tilde{p}(0)^{-1}$ (this is required in order that there is no steady-state offset).

The set of allowable processes, Π, in Eqs. 2 and 3 can be described in either of the following ways:

(S1) The set Π is the set of all transfer functions $p(s, \alpha, \beta(\alpha))$ for stable or unstable processes, p, with $\underline{\alpha}_i \leq \alpha_i \leq \overline{\alpha}_i$, and β a continuous function of α, satisfying

$$\left.\begin{array}{ll} p(0,\alpha,\beta(\alpha))>0 \text{ for all } \alpha & \text{These restrictions ensure that all} \\ p(0,\alpha,\beta(\alpha))<0 \text{ for all } \alpha. & \text{process gains have the same sign} \end{array}\right\}$$

The function $\beta(\alpha)$ is introduced to allow for correlated parameters.

(S2) The set Π is the set of all transfer functions $p(s)$ for stable or unstable processes, p, such that,

$$\underline{M}(\omega) \leq |p(i\omega)| \leq \overline{M}(\omega)$$

$$\underline{A}(\omega) \leq \operatorname{Arg} p(i\omega) \leq \overline{A}(\omega),$$

where $\overline{M}(\omega)$ and $\underline{M}(\omega)$ are upper and lower bounds on the magnitude of all possible process frequency responses, and $\overline{A}(\omega)$ and $\underline{A}(\omega)$ are upper and lower bounds on the phase angle of all possible process frequency response. As in the set definition given by S1, we again require that the steady-state gain of all processes in Π have the same sign. In terms of the frequency response this requirement translates as

$$\overline{A}(0) = \underline{A}(0) = 0°$$

or

$$\overline{A}(0) = \underline{A}(0) = 180°.$$

The uncertainty description given by S2 is the one most commonly found in the literature, usually without bounds on the phase angle of the transfer function. However, since such bounds are even more difficult to obtain for real processes than the parametric bounds given by S1, we will restrict our examples to parametric uncertainty descriptions.

Example 1: Mp-Tuning of an Uncertain First-Order Lag Plus Dead-Time Processes. The process description is

$$p(s) = \frac{Ke^{-Ts}}{(\tau s+1)} \quad K \in [0.2,1]; \tau \in [8,14]; T \in [0.4,1]. \quad (5a)$$

Selecting a midrange model, and an IMC controller that inverts this model, yields

$$\tilde{p}(s) = \frac{0.6e^{-0.7s}}{(11s+1)} \quad (5b)$$

$$q(s) = \frac{(11s+1)}{0.6(\epsilon s+1)}. \quad (5c)$$

Our aim in tuning is to find an IMC controller filter time constant, ϵ, so that no closed-loop response to a step setpoint change overshoots the desired setpoint by more than 10% for all processes in the uncertainty set. Further, we wish to estimate the range of closed-loop response times for all processes in the uncertainty set.

To achieve a "worst case" overshoot of 10% in the time domain, we seek a filter time constant, ϵ, that yields a maximum magnitude of the complementary sensitivity function of 1.05 (cf. Figure 2 and Eqs. 2 and 3. This choice for the Mp is motivated by the fact that a second-order system that has a maximum peak of 1.05 has a 10% overshoot to a step change in input. Even though most closed-loop systems are not second order, including our current example, they often exhibit the same overshoot−Mp relationship as for second-order systems.

The upper and lower bounds on the maximum magnitude of the complementary sensitivity function are given in Figure 3, where the upper bound has an Mp of 1.05. The worst-case process (that is, the process parameters that yield an Mp of 1.05) is $[K=1, \tau=8, T=1]$. The gain and dead times are at their upper limits, while the time constant is at its lower limit.

In Figure 3, the upper-bound break frequency is about 1.1, while the lower-bound break frequency is about 0.072. The time constants for the fastest and slowest responses are roughly one over the break frequencies, or 0.9 and 14, respectively. Estimating the settling times as three time constants yields a range of response times between 2.7 and 42 time units. These estimates are in good agreement with the actual time responses shown in Figure 4. Also notice that the maximum overshoot is about 10%, as desired.

The processes associated with the upper and lower bound curves in Figure 3 are those that yield the maximum or minimum of the complementary sensitivity function at a particular frequency. For example, Table 1 lists the processes associated with the lower bound of Figure 3. Notice that the same process parameters yield the lower bound over portions of the frequency domain.

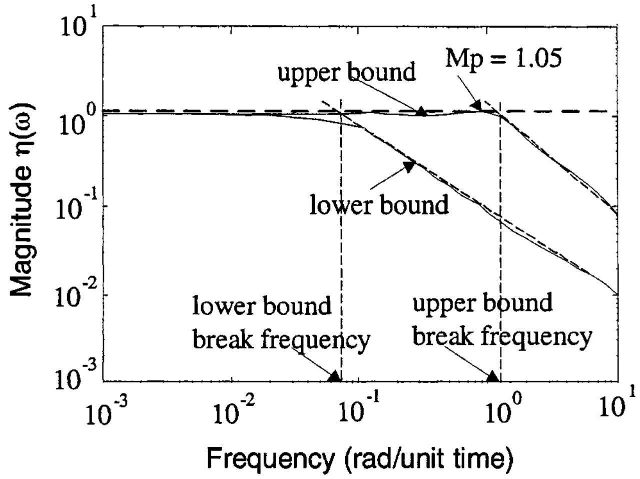

Figure 3. Upper and lower bound of complementary sensitivity functions when $\epsilon = 2.81$ for Example 1.

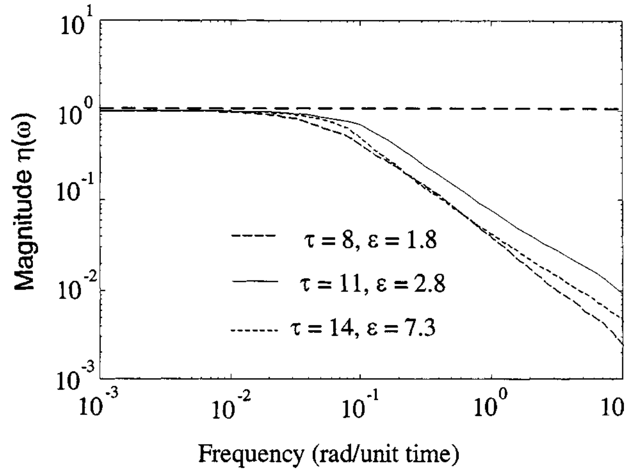

Figure 5. The lower bound of the complementary sensitivity function for different IMC controller lead time constants.

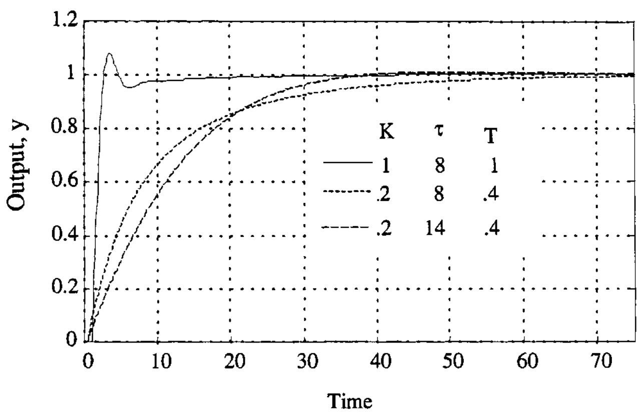

Figure 4. Time responses of various processes for Example 1 to a setpoint step change ($\epsilon = 2.81$).

Example 2: Comparison of Alternate Control-System Designs for Example 1. Selecting a midrange controller and model in Example 1 seems reasonable, but might other choices give better performance? For this example it appears not. Changing the model and controller gains to one, and retuning to get an Mp of 1.05 yields a filter time constant of 1.30. However, the upper- and lower-bound curves are insignificantly different from those in Figure 3. Similarly, the time responses are virtually the same as those in Figure 4.

Figure 5 compares the lower bound of the complementary sensitivity function for different controller lead time constants (that is, values different from 11 in Eq. 5c). The model remains the same as Eq. 5b, and each control system is tuned to achieve an Mp of 1.05.

Based on the frequency responses of Figure 5 the speed of response of the slowest closed-loop response is not improved by using a different controller lead time constant. Indeed, the slowest responses are actually slightly slower for the alternate controllers. Time-domain simulations (not shown) confirm this conclusion.

Figure 6 compares the lower bound of the complementary sensitivity function for different controller lead and model time constants (that is, values different from 11 in both Eqs. 5b and 5c). The model lag time constant, and the controller lead time constant are the same. Again, there is no advantage to using other than the midrange model and controller.

Figures 3 through 6, and the data in Table 1 were all obtained with the aid of a suite of Matlab programs that we call IMCTUNE. This software is described in Appendix A, along

Table 1. Processes Associated with the Lower Bound in Figure 3

Frequency Range	K	τ	T
0.001–0.10	0.2	8	0.4
0.10–10	0.2	14	0.4

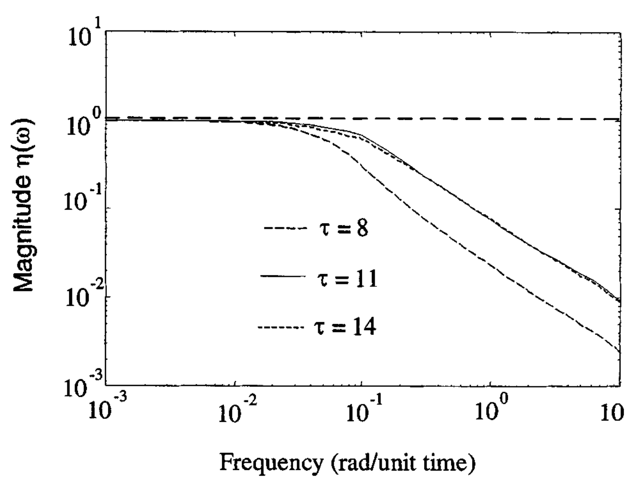

Figure 6. The lower bound of the complementary sensitivity function for different IMC controller lead and model lag time constants.

with the algorithm used to compute the IMC filter time constant that satisfies a given Mp specification. The software is available without charge at the Web Site of the last named author (C. Brosilow) at Case Western Reserve University (the current URL is: http://cheme.cwru.edu/People/Faculty/brosilow/brosilow.htm#brosilow, but the reader should recognize that URLs change to accommodate new situations).

As an example of the use of IMCTUNE, we used it to get the PID controller approximation to the feedback controller, $c(s)$, (c.f. Figure 1) formed by collapsing the feedback around the IMC controller, q, and the model, $\tilde{p}$, in Figure 2. That is,

$$c(s) \equiv q(s)/(1-\tilde{p}(s)q(s)) \cong K_c(1+1/(\tau_I s)+\tau_d s/(0.05\tau_d s+1)). \qquad (6)$$

IMCTUNE uses the approach of Lee et al. (1998), and gives $K_c = 5.26$, $\tau_I = 11.1$, $\tau_D = 0.068$. This PID controller, in the standard feedback loop of Figure 1, gives the same responses as the IMC control system. Thus, by tuning an IMC controller, we also obtain a well-tuned PID controller.

In the foregoing, we have implicitly assumed that we can, by choosing the filter time constant large enough, achieve any Mp specification greater than one. In general this assumption is false unless the model gain is chosen appropriately. The next subsection gives the conditions under which it is possible to find a controller filter time constant, ϵ, for any Mp specification greater than 1.

Dependence of the minimum achievable Mp on the model gain and process gain uncertainty range

Table 2 shows how the maximum value of the process gain to the model gain (that is, $p(0)/\tilde{p}(0)$) varies with controller filter order. This table was obtained as described in Appendix B.

From Table 2 we see that it may not be possible to choose a midrange value for the gain of the nominal model when the relative order is 3 or higher. That is, if

$$\tilde{p}(0) = \frac{\max p(0) + \min p(0)}{2},$$

then the maximum value for $p(0)/\tilde{p}(0)$ is 2. A value of $p(0)/\tilde{p}(0)$ of 2 with a filter of relative order of 3, leads to a value for the smallest maximum peak of $|\eta|$ of 1.094. Of course, it is always possible to choose the nominal model gain to be sufficiently greater than the midrange gain so that the maximum value of $p(0)/\tilde{p}(0)$ is less than 1.5. Therefore, it is always possible to specify any Mp greater than one, and the specified Mp will be achieved for a sufficiently large filter time constant.

Table 2. Upper Limits on the Ratio of Process to Model Gain that Permit Any Mp Specification Greater than One to be Achieved by Selecting a Large Enough Filter Time Constant

Filter Order, n	Upper Limit of the Gain Ratio $p(0)/\tilde{p}(0)$
1	∞
2	2
3	1.5
4	1.34
5	1.25

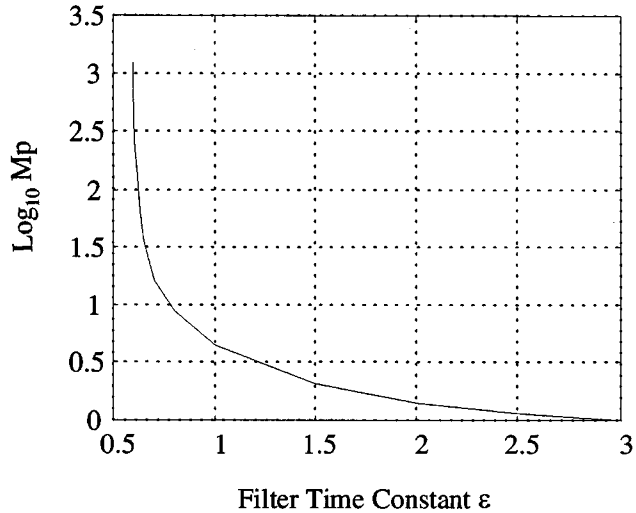

Figure 7. Behavior of Mp with filter time constant for Example 1.

The possibility of multiple solutions of the Mp tuning problem and what to do about it

Figure 7 shows the behavior of Mp (that is, the peak of the maximum magnitude of the complementary sensitivity function) as a function of the IMC controller filter time constant for Example 1. As expected from the Robust Stability theorem, the peak approaches infinity as the filter time constant is reduced and the control system becomes unstable for one or more processes in the uncertainty set. Considerable numerical experience indicates that Figure 7 is typical of the behavior of Mp with a filter time constant for closed-loop control of inherently stable, infinite-dimensional processes such as that of example 1.

Figure 7 is probably also typical of processes modeled as finite-dimensional systems provided that the control system becomes unstable for small controller filter time constants. However, it is possible to approximate a process with a model that doesn't allow the control system to become unstable for any positive filter time constant. In that case, the behavior of Mp with filter time constant need not follow the nice monotone behavior shown in Figure 7, but rather can be like that of Figure 8.

From Figure 8, an Mp of 1.05 requires a filter time constant of 31. At that point the maximum magnitude is smoothly decreasing as the filter time constant increases. If the specification were for an Mp of 1.1, however, then there would be multiple values of filter time constants (1.5, 2, and 14) that satisfy this specification. For such a situation, we suggest choosing a value filter time constant so that the Mp decreases monotonically with increasing filter time constant. From Figure 8, we would choose a value for ϵ of 14 to achieve an Mp of 1.1. For filter time constants of 1.5 and 2, the Mp increases for modest increases in filter time constant. This implies that if the control system slows down for any reason, such as control effort saturation, the process output may exhibit more overshoot than desired.

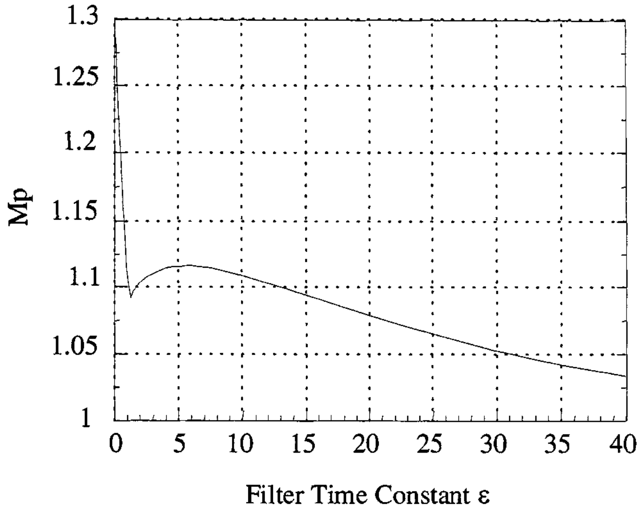

Figure 8. Dependence of maximum peak of $|\eta(i\omega)|$ on the filter time constant for the process: $p(s) = K(\tau_N s+1)/(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)$; $K\epsilon[5,15]$, $\tau_N\epsilon[32,40]$, $\tau_1\epsilon[10,20]$, $\tau_2\epsilon[40,50]$, $\tau_3\epsilon[2,6]$; the model: $\tilde{p}(s) = 10(36s+1)/(15s+1)(45s+1)(4s+1)$; the controller: $q(s) = (15s+1)(45s+1)(5s+1)/10(36s+1)(\epsilon s+1)^2$.

An Mp synthesis example

Mp tuning provides upper and lower bounds on the closed-loop time constants for a fixed-model and filter time constant. Mp synthesis addresses the problem of finding the best model and controller. The optimal model and controller are those that shift the lower-bound break frequency furthest to the right without increasing the Mp, thereby speeding up the slowest closed-loop responses. In the following example we explore the effect of changing just the controller and then both the controller and the model. We shall see that, unlike Example 2, the effects can be substantial for processes whose relative order is greater than one where is even a modest uncertainty in the process parameters.

Example 3. An over damped second-order plus dead-time process:

Process:

$$p(s) = \frac{K}{(8s+1)(6s+1)} e^{-Ts} \quad 5 \le K, T \le 15. \tag{7a}$$

Model:

$$\tilde{p}(s) = \frac{10}{(8s+1)(6s+1)} e^{-10s}. \tag{7b}$$

Inverse midrange model controller:

$$q(s) = \frac{(8s+1)(6s+1)}{10(28.6s+1)^2}. \tag{7c}$$

Inverse approximate midrange model controller:

$$q(s) = \frac{(14s+1)}{10(31.6s+1)}. \tag{7d}$$

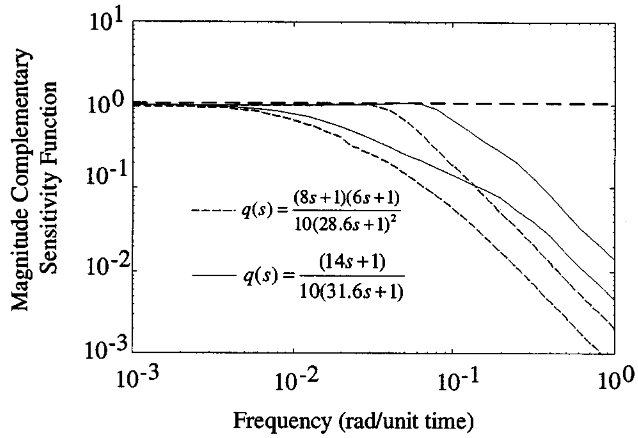

Figure 9. Comparison of the upper and lower bounds of the closed-loop frequency responses for the system of Eq. 9.

The controller given by Eq. 7c is the normal inverse of the invertible part of the model multiplied by a second-order filter to make the controller realizable. The controller given by Eq. 7d is the inverse of an approximate model with the original second-order lag replace by a first-order lag whose time constant is the sum of the time constants of the second-order lag. Both controllers have been tuned to give an Mp of 1.05.

Figure 9 gives the upper- and lower-bound curves of the complementary sensitivity function for control systems that use the controllers given by Eqs. 7c and 7d. The model is that of Eq. 7b for both control systems. Notice that both the upper- and lower-bound responses for the controller given by Eq. 7d are shifted to the right of those for the controller given by Eq. 7c, implying that the fastest and slowest time-domain responses for the controller given by Eq. 7d will be faster than those for Eq. 7c. Figure 10 bears this out.

Changing the model given by Eq. 7b to

$$\tilde{p} = \frac{10e^{-10s}}{(14s+1)},$$

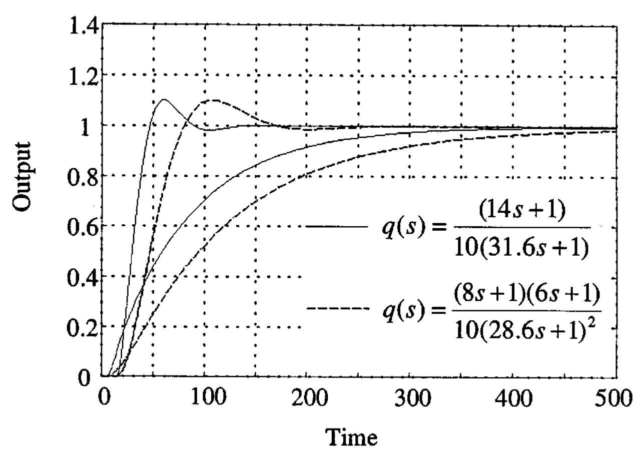

Figure 10. Fastest and slowest responses (that is, $K = 15$ and 5, respectively) to a step setpoint change for the control system of Eq. 9.

and using the controller given by Eq. 7d changes the control system responses insignificantly from those given in Figures 9 and 10.

Standard engineering practice for the control of over damped systems such as that given by Eq. 7 is to fit a first-order model of the form in Eq. 8a to the output response to a step change in the control effort, and then to tune the controller based on that model (Seborg et al., 1989, p. 170, 302 ff):

$$\tilde{p}(s) = \frac{K}{(\tau s + 1)} e^{-(T+\Delta)s}, \tag{8a}$$

where
T = the estimated process dead time
Δ = an additional dead time arising from the fitting procedure
τ = an estimate of a first-order time constant that allows the approximate model step response to match that of the higher-order process.

Using the reaction-curve method (Seborg et al., 1989, p. 170) yields $(T + \Delta) = 12$, $\tau = 20$, and, of course, $K = 10$. Tuning the control system using this model and its model inverse controller yields a filter time constant of 38.6. The resulting controller is:

$$q = \frac{(20s+1)}{10(38.6s+1)}. \tag{8b}$$

The model and a controller given by Eqs. 8a and 8b does not perform quite as well as the controller and model given by Eqs. 7b and 7d. It turns out that the preceding model parameters do not give a very good fit to a step change in the control effort. A much better fit is obtained with $\Delta = 4$, and $\tau = 11$. The responses with this model and its associated controller are insignificantly different from those of Eqs. 7b and 7d (that is, the solid line responses in Figure 10).

The preceding models and controllers are all "midrange" in the sense that the model gain is the midrange gain, and the dead time is either the midrange dead time, or the dead time obtained by fitting the midrange model. However, one can obtain a performance that is slightly better than that obtained with Eqs. 7b and 7d with either of the following models and controllers:

$$\tilde{p}(s) = \frac{15}{(8s+1)(6s+1)} e^{-15s} \quad q(s) = \frac{(8s+1)(6s+1)}{15(6.07s+1)^2} \tag{9}$$

$$\tilde{p}(s) = \frac{15}{(8s+1)(6s+1)} e^{-15s} \quad q(s) = \frac{(14s+1)}{15(11s+1)}. \tag{10}$$

The two systems just given yield the same frequency and time responses, and these are given in Figures 11 and 12, respectively.

Notice that in Figure 11 the Mp is one, and the criterion that establishes the controller filter time constant is that no peak-to-valley height be greater than 0.1. The process at the local peak in Figure 11 is one with $K = 15$, $T = 6.3$, and this process yields the somewhat oscillatory response shown in Figure 12. Also, comparison of the responses of midrange process (that is, $K = 10$, $T = 10$) with those of Eqs. 7b and 7d, where the model is perfect shows no significant difference. *Thus there is no apparent advantage to using the midrange model, even if the process operates around the midrange far more often than at K = 15, T = 15.*

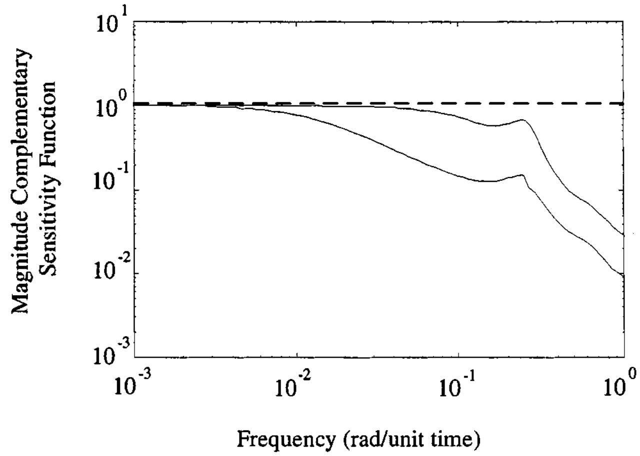

Figure 11. Upper- and lower-bound frequency responses for the control systems given in Eqs. 11 and 12.

Changing the model in Eq. 10 to $\tilde{p}(s) = 15e^{-15s}/(14s+1)$ and retuning gives $q(s) = (14s+1)/15(10.6s+1)$. This controller model pair has an Mp of 1.05, and the lower-bound curve of the complementary sensitivity function actually lies slightly to the right of that in Figure 11. Once again, the simpler model/controller is preferred.

As a check on the sensitivity of the control systems just discussed (that is, Eqs. 7b and 7d) to the assumed uncertainty bounds, we increased the upper and lower bounds by $\pm 20\%$ to $\overline{K}$, $\overline{T} = 18$, and $\underline{K}$, $\underline{T} = 4$. This change causes worst-case overshoots of 30% for Eqs. 9 and 10 and a 35% overshoot for Eqs. 7b and 7d. The settling time of the slowest responses increases to 500 and 450 units, respectively. Thus, the control system is not overly sensitive to modest errors in estimating the parameter uncertainty ranges.

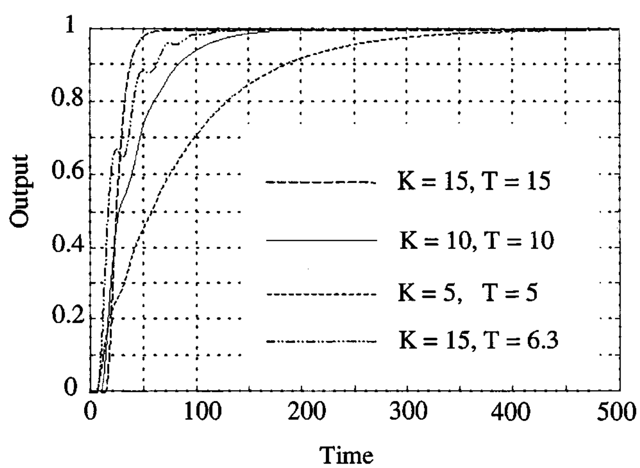

Figure 12. Time responses of the control system given by Eqs. 11 and 12.

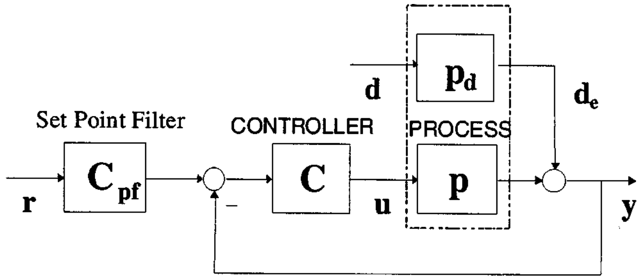

Figure 13. Standard-feedback two-degrees-of-freedom control system.

The PID Controllers, which have the same performance as Eqs. 7b, 7d, 9, and 10, are, respectively (cf. Eq. 6), $K_c = 0.0338$, $\tau_I = 14.0$, $\tau_D = 1.73$ for Eqs. 7b and 7d; $K_c = 0.0422$, $\tau_I = 16.5$, $\tau_D = 4.42$ for Eq. 9; and $K_c = 0.0412$, $\tau_I = 16.8$, $\tau_D = 4.41$ for Eq. 10.

Design and Tuning of Two-Degrees-of-Freedom Controllers

The structure and preliminary design of two-degrees-of-freedom control systems

The two-degrees-of-freedom control structures of Figures 13 and 14 can be useful whenever the disturbance enters the process output through a lag [that is, $p_d(s)$], whose effective time constant is significantly larger than that of the process [that is, $p(s)$], or the process is unstable.

The closed-loop transfer functions between the setpoints and the disturbances and the process output for Figures 13 and 14 are

$$y(s) = \frac{c_{pf}(s)c(s)p(s)r(s) + p_d(s)d(s)}{(1 + c(s)p(s))} \quad \text{for Figure 13} \tag{11}$$

$$y(s) = \frac{q(s,\epsilon_r)p(s)r(s) + (1 - \tilde{p}(s)qq_d(s,\epsilon))p_d(s)d(s)}{(1 + (p(s) - \tilde{p}(s))qq_d(s,\epsilon))} \quad \text{for Figure 14.} \tag{12}$$

The transfer functions given by Eqs. 11 and 12 are equivalent (in terms of input−output transfer functions) if $c_{pf}(s)$ and $c(s)$ are chosen as

$$c_{pf}(s) = q(s,\epsilon_r)(qq_d(s,\epsilon))^{-1} \tag{13}$$

$$c(s) = \frac{qq_d(s,\epsilon)}{(1 - \tilde{p}(s)qq_d(s,\epsilon))}. \tag{14}$$

The two-degrees-of-freedom control system reduces to that of a single-degree-of-freedom control system (cf. Figures 1 and 2) if $qq_d(s,\epsilon)$ is chosen as $q(s,\epsilon)$ and ϵ_r is selected as ϵ.

The objective of a two-degrees-of-freedom control system for a stable process is to speed the response to disturbances while not degrading the setpoint response. The simplest controller design for speeding up the response to disturbances is to choose $qq_d(s,\epsilon)$ as $q(s,\epsilon)$, but to select ϵ to be smaller than that which would achieve an Mp of, say, 1.05 for a one-degree-of-freedom control system. Smaller filter time constants are feasible because the filtering action of the disturbance lag, $p_d(s)$, reduces the amplitude of the disturbance frequencies that enter the feedback loop. This, in turn, reduces the magnitude of loop oscillations and allows the selection of a smaller filter time constant, thereby speeding up the disturbance response. The setpoint controller filter, ϵ_r, can then be chosen to achieve any desired Mp.

A potentially better design than this is to choose $qq_d(s,\epsilon)$ so that the zeros of $(1 - \tilde{p}(s)qq_d(s,\epsilon))$ cancel one or more poles $p_d(s)$. An easy way of accomplishing this task is to take the controller $qq_d(s,\epsilon)$ as being composed of two terms, $q(s,\epsilon)$ and $q_d(s,\epsilon)$. The term $q(s,\epsilon)$ is designed as for a one-degree-of-freedom control system, and the numerator polynomial coefficients, α_i, of $q_d(s,\epsilon)$ in Eq. 15 below, are selected to make zeros of $(1 - \tilde{p}(s)qq_d(s,\epsilon))$ cancel selected poles in $p_d(s)$:

$$q_d(s,\epsilon,\alpha) = \frac{\sum_{i=0}^{n} \alpha_i s^i}{(\epsilon s + 1)^n}, \tag{15}$$

where n is number of poles in $\tilde{p}_d(s)$ to be canceled by the zeros of $(1 - \tilde{p}(s)qq_d(s))$.

Mp tuning and synthesis of two-degrees-of-freedom control systems for stable processes

To tune the feedback loop of Figure 14 for good disturbance suppression, it seems reasonable to focus on the disturbance to output transfer function, called the sensitivity function, which is given by (cf. Eq. 12):

$$\text{Sensitivity function} \equiv \frac{y(s)}{d(s)} = \frac{(1 - \tilde{p}(s)qq_d(\epsilon,s))p_d(s)}{(1 + (p(s) - \tilde{p}(s))qq_d(\epsilon,s))}. \tag{16}$$

The controller design and tuning most commonly found in the literature (Morari and Zafiriou, 1989) is to seek a controller that achieves a specified, frequency-dependent upper bound on the magnitude of the preceding sensitivity function.

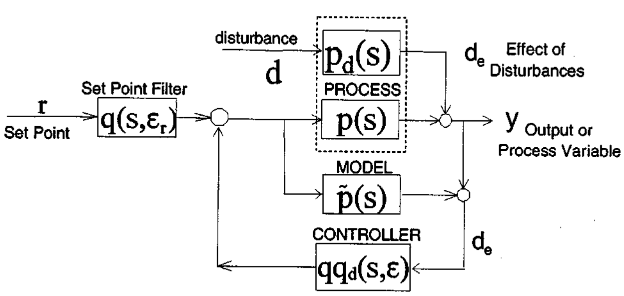

Figure 14. IMC form of two-degrees-of-freedom control system.

Unfortunately, it takes a good deal of experience to specify such an upper bound so as to achieve the desired time-domain disturbance rejection. The reason is that, unlike the complementary sensitivity function, there is no fixed upper bound on the peak of the sensitivity function that corresponds to a desirable closed-loop time performance for a large number of process transfer functions. Because it is not a simple matter to set bounds on the magnitude of the sensitivity function so as to achieve desirable time-domain behavior, we next introduce a modification of the sensitivity function that is easier to use.

The first term of the sensitivity function in Eq. 16, $p_d(s)/(1+(p(s)-\tilde{p}(s))qq_d(\epsilon, s))$, is the output response to the disturbance, as modified by the closed loop. The second term, $-\tilde{p}(s)qq_d(\epsilon, s)p_d(s)/(1+(p(s))qq_d(\epsilon, s))$, represents the response of the output to the control effort. This term is the negative of the model output response to a setpoint that is filtered by (that is, cascaded with) the disturbance lag, $p_d(s)$. Replacing the model, $\tilde{p}(s)$, with $p(s)$ and taking the IMC controller to be $qq_d(s, \epsilon)$ makes the term the same as the complementary sensitivity function filtered by the disturbance lag. This observation suggests the possibility of applying the Mp tuning algorithm to the complementary sensitivity function filtered by the disturbance lag. Since a descriptive name such as ''filtered complementary sensitivity function'' is a bit of a mouthful, we shorten it to simply the ''partial sensitivity function.''

$$\text{Partial sensitivity function} \equiv \frac{p(s)qq_d(\epsilon,s)p_d(s)/p_d(0)}{(1+(p(s)-\tilde{p}(s))qq_d(\epsilon,s))}. \tag{17}$$

The disturbance lag in Eq. 17 is divided by its gain so that the steady-state gain of the partial sensitivity function will be one, and it can therefore be used for tuning just like the complementary sensitivity function. Since the magnitude of the disturbance is generally unknown, there is no loss in generality by this normalization. The following examples demonstrate the use of the partial sensitivity function.

Example 4. This example is the same as Example 3, except that the disturbance passes through the process. That is,

$$p(s)=\frac{K}{(8s+1)(6s+1)}e^{-Ts} \quad p_d(s)=\frac{1}{(8s+1)(6s+1)} \qquad 5 \le K, T \le 15. \tag{18}$$

In the disturbance lag, $p_d(s)$, in Eq. 18, the gain has been normalized to one, and the dead time removed, since neither the magnitude of the disturbance nor the time that it enters the process are known *a priori*, nor are they relevant to the controller design.

Based on the discussion following Example 3, we select the process model as

$$\tilde{p}(s)=\frac{15}{(8s+1)(6s+1)}e^{-15s}. \tag{19}$$

Selecting $q(s, \epsilon)$ to invert $15/(8s+1)(6s+1)$, and $q_d(s, \epsilon)$ as given by Eq. 15 yields

$$qq_d(s,\epsilon)=\frac{(8s+1)(6s+1)^2(7.96s+1)}{15(5.64s+1)^4}, \tag{20}$$

where $\epsilon = 5.64$. Selecting $q_d(s, \epsilon)$ as one and tuning gives

$$qq_d(s,\epsilon)=\frac{(8s+1)(6s+1)}{15(4.36s+1)^2}, \tag{21}$$

where $\epsilon = 4.36$.

The partial sensitivity function upper- and lower-bound magnitude curves for the controllers given by Eqs. 20 and 21 and the model given by Eq. 19 are virtually identical, and very similar to that of Figure 11. Further, they are similar to those for the controller given by Eq. 9 (which is the same as Eq. 21 with $\epsilon = 6.07$), but lie slightly to the left of it. Thus the disturbance responses of the control system employing Eq. 19 and either Eq. 20 or Eq. 21 show slightly faster recovery from step disturbances passing through $(1/(8s+1)(6s+1))$. However, the differences in performance are so small as to be not worth showing. This situation doesn't change if the uncertainty limits are reduced to $9 \le K,\ T \le 11$. That is, while all the responses can be sped up, there is still no significant benefit to using a two-degrees-of-freedom controller over a single-degree-of-freedom controller. The situation changes somewhat, however, if the disturbance lag is twice or more as large as the lag between the output and the control effort as in Example 5 below.

Example 5.

$$p(s)=\frac{K}{(8s+1)(6s+1)}e^{-Ts}, \quad p_d(s)=\frac{1}{(16s+1)(12s+1)} \qquad 5 \le K, T \le 15 \tag{22}$$

$$\tilde{p}(s)=\frac{15}{(8s+1)(6s+1)}e^{-15s}. \tag{23}$$

Selecting $qq_d(s, \epsilon)$ so that the zeros of $(1-\tilde{p}(s)qq_d(s, \epsilon))$ cancel the poles of $p_d(s)$ in Eq. 22 gives

$$qq_d(s,\epsilon)=\frac{(8s+1)(6s+1)(12.3s+1)(14.2s+1)}{15(7.89s+1)^4}. \tag{24a}$$

Selecting $q_d(s, \epsilon)$ as one and returning to obtain ϵ gives

$$qq_d(s,\epsilon)=\frac{(8s+1)(6s+1)}{15(3.85s+1)^2}. \tag{25a}$$

The lower-bound curves for the two-degrees-of-freedom control systems using the preceding controllers lie appreciably to the right of that for the single-degree-of-freedom control system, as shown by Figure 15. Further, Eq. 24 yields the best performance. The associated slowest time responses are given in Figure 16.

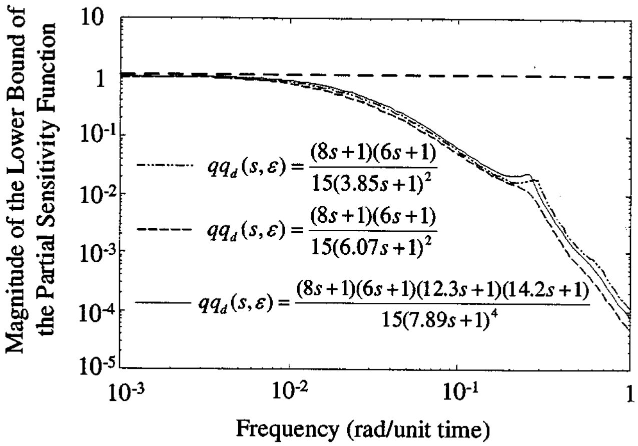

Figure 15. Comparison of single- and two-degrees-of-freedom controllers for Example 5.

The setpoint controllers associated with Eqs. 24 and 25 (cf. Figure 14) are

$$q(s,\epsilon_r)=\frac{(8s+1)(6s+1)}{15(24.9s+1)^2} \tag{24b}$$

$$q(s,\epsilon_r)=\frac{(8s+1)(6s+1)}{15(23.6s+1)^2}. \tag{25b}$$

Responses to a step setpoint change using these setpoint "filters" are essentially the same as those shown in Figure 12.

The PID controllers and setpoint filters (cf. Figure 13) derived from Eqs. 24 and 25 are, respectively,

$$c(s)=0.0545\left(1+\frac{1}{16.38s}+\frac{5.536}{(0.277s+1)}\right)$$

$$c_{pf}(s)=\frac{(7.88s+1)^4}{(12.3s+1)(14.2s+1)(24.9s+1)^2} \tag{24c}$$

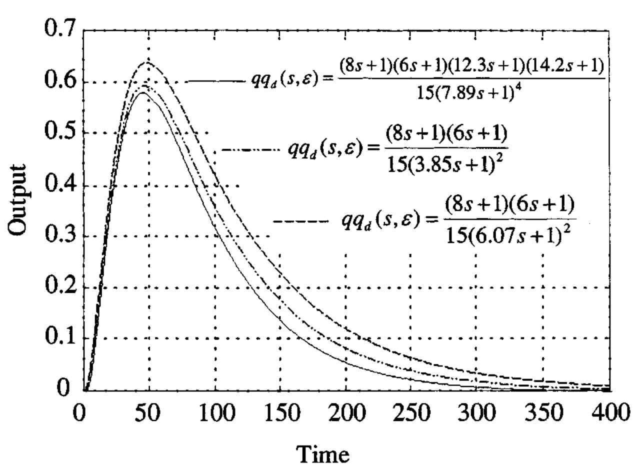

Figure 16. Comparison of single- and two-degrees-of-freedom controllers for Example 5.

$$c(s)=0.0538\left(1+\frac{1}{18.3s}+\frac{5.57}{(0.279s+1)}\right)$$

$$c_{pf}(s)=\frac{(3.85s+1)^2}{(23.6s+1)^2}. \tag{25c}$$

The modest gain in performance achieved by the two-degrees-of-freedom control system comes at the price of a slightly more complex implementation and at a somewhat diminished stability margin. To understand the diminished stability margin, compare the filter time constants of the two-degrees-of-freedom controller given by Eq. 25 (that is, $\epsilon=3.85$), and that given by Eq. 9 (that is, $\epsilon=6.07$). The two-degrees-of-freedom controller can have a smaller filter time constant because of the filtering action of the disturbance lag and the setpoint filter. However, both the setpoint filter and the disturbance lag are outside of the feedback loop. The relative stability of the feedback loop is given by the Mp of the partial sensitivity function with the disturbance lag set to one (that is, $p_d(s)=1$). In this case, the partial sensitivity function and the complementary sensitivity function are identical, and the Mp is 3.31. Similarly, the Mp of Eq. 24, with $p_d(s)=1$, is 3.29. Certainly both control systems are still far from instability. If the disturbance lags were substantially larger than those of this example, however, then a two-degrees-of-freedom design might result in a control system with a very small stability margin. In this case, we suggest that the design criterion should shift from the partial sensitivity function back to the complementary sensitivity function, but now with a much higher Mp specification (such as Mp = 10).

Mp Tuning and Synthesis of Two-Degrees-of-Freedom Control Systems for Unstable Processes

The IMC two-degrees-of-freedom control system of Figure 14 cannot be used to implement control of an inherently unstable process because the configuration is itself internally unstable for an unstable process (Morari and Zafiriou, 1989). However, Figure 14 can be used for the design and tuning of an IMC control system, which is then implemented via Figure 13 using the relationships given by Eqs. 13 and 14. Alternately, the algorithms proposed by Berber and Brosilow (1997, 1999) and Cheng and Brosilow (1987) can implement the IMC controllers. These yield the same input–output transfer functions as in Figure 14, but are internally stable. To illustrate the foregoing, we present two examples. The first is an undamped integrating process drawn from the literature. Our interest in this example stems mainly from the fact that the literature solution provides an independent check on the efficacy of the Mp tuning and synthesis method of this article. The second example is that of an unstable first-order lag and dead time that is more typical of chemical processes.

Example 6. This example presents the solution for an unstable "benchmark" control problem proposed by Wie and Bernstein (1990, 1992). It consists of an undamped two-mass spring system modeled by

$$y(s)=p(s)u(s)+p_d(s)d(s) \tag{26a}$$

$$p(s) = \frac{\frac{1}{2}}{s^2\left(\frac{1}{2k}s^2+1\right)}; \quad p_d(s) = \frac{\frac{1}{2}\left(\frac{1}{k}s^2+1\right)}{s^2\left(\frac{1}{2k}s^2+1\right)}$$

$$0.5 \le k \le 2, \quad (26b)$$

where
k = process spring constant, which varies between 0.5 and 2
u = control effort
d = disturbance
y = position of the second mass.

The control objective is to design a robust controller to suppress impulse disturbances for all the plants in the uncertainty set. Further, the "nominal" process is to have a settling time of 15 s.

As before, we take the controller $qq_d(s,\epsilon)$ as being composed of two terms, $q(s,\epsilon)$ and $q_d(s,\epsilon)$. The term $q(s,\epsilon)$ is taken as the inverse of the model of the process as given below:

$$q(s,\epsilon) = \frac{s^2\left(\frac{1}{2\tilde{k}}s^2+1\right)}{\frac{1}{2}(\epsilon s+1)^4}, \quad (27)$$

where
$\tilde{k}$ = model spring constant
ϵ = adjustable controller filter time constant.

The preceding choice for $q(s,\epsilon)$ reduces the problem to that of choosing $q_d(s,\epsilon)$ so that the zeros of $(1-q_d(s,\epsilon)/(\epsilon s+1)^4)$ cancel the poles of $\tilde{p}_d(s)$. This requires that $q_d(s,\epsilon)$ be at least third order, as given by Eq. 28.

$$q_d(s) = \frac{\tau_3 s^3+\tau_2 s^2+\tau_1 s+1}{(\epsilon s+1)^3}. \quad (28)$$

The constants τ_1, τ_2, and τ_3 in Eq. 28 are chosen so that the numerator of $(1-q_d(s,\epsilon)/(\epsilon s+1)^4)$ has zeros at $s=0$ and at $s=\pm i\sqrt{2\tilde{k}}$. There is automatically an additional zero at $s=0$ by virtue of the fact that $q_d(0,\epsilon)$ is one. Equating coefficients of the requisite polynomials gives

$$\tau_1 = 7\epsilon \quad (29a)$$

$$\tau_2 = 21\epsilon^2 - 70\tilde{k}\epsilon^4 + 28\tilde{k}^2\epsilon^6 \quad (29b)$$

$$\tau_3 = 35\epsilon^3 - 42\tilde{k}\epsilon^5 + 4\tilde{k}^2\epsilon^7 \quad (29c)$$

These constants depend on the value of the filter time constant and the model parameter $\tilde{k}$. The filter time constant, ϵ, is selected to be 1 so that the nominal plant with $\tilde{k}=1$ has the desired settling time of about 15 s, as required in the benchmark problem specifications (Wie and Bernstein, 1990, 1992). Having thus chosen ϵ, our only remaining task is to find an optimal value for $\tilde{k}$.

We will choose that value of $\tilde{k}$ that minimizes the maximum peak sensitivity function, Eq. 16. By exploring values of the spring constant, $\tilde{k}$, between 0.5 and 2, we determined that the minimum peak is obtained using a model spring constant of 0.7. The feedback controller for this spring constant from Eq. 14 is given by

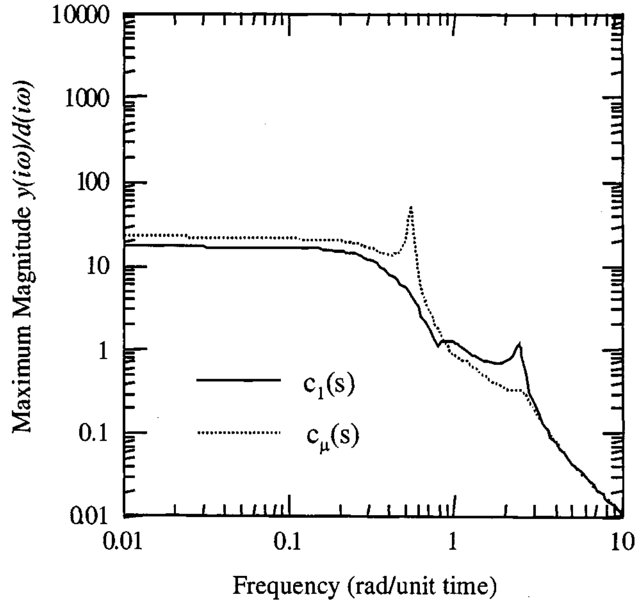

Figure 17. Maximum of the magnitude of the sensitivity function using controllers $c_1(s)$ and $c_\mu(s)$.

$$c_1(s) = \frac{10.8(s^3-1.88s^2+0.93s+0.13)}{s^3+7s^2+19.6s+25.2}. \quad (30)$$

Values for $\tilde{k}$ of near 0.5 and 2 yield unstable control systems that are indicated by very large peaks in the maximum of the sensitivity function.

Braatz and Morari (1992) solved the preceding control problem using the *D-K* iteration method (Doyle, 1985) and loop shaping, and obtained the following "μ-optimal" controller:

$$c_\mu(s) = \frac{0.0443(9.402s+1)(-2.697s+1)(0.4789s+1)}{(0.216s^2+0.861s+1)(0.118s^2+0.369s+1)}. \quad (31)$$

Figure 17 shows that the maximum magnitude of the sensitivity function is bounded for both control systems. Since a separate Nyquist analysis of the control systems for a process with a spring constant of $k=0.8$ shows that each control system is stable for that value of spring constant, we conclude that the control system is stable for all values of spring constant in the uncertainty set. Further, since the maximum of the sensitivity function over all frequencies is smaller for controller $c_1(s)$ than for controller $c_\mu(s)$, we expect that time responses for the control system using $c_1(s)$ are likely to be less oscillatory than that for $c_\mu(s)$, at least for some values of the spring constant. The time responses in Figures 18 and 19 confirm this expectation.

Example 7. This example demonstrates the application of the Robust Stability theorem to an infinite dimensional system. It also points out the need to check the stability of one process that is not equal to the model in order to draw correct conclusions regarding the stability of all processes in the set of uncertain processes.

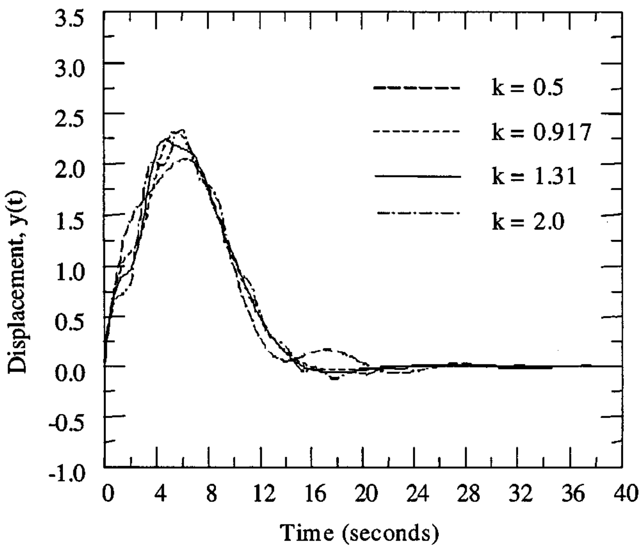

Figure 18. Response of $y(t)$ to an impulse disturbance for different spring-constant values using controller $c_1(s)$.

The process is

$$p(s) = \frac{Ke^{-s}}{(-s+1)} \qquad 0.9 \le K \le 1.1. \tag{32}$$

The preceding process is one that is not easy to control, in spite of the relatively small range of uncertain gains, because of the relatively large dead time to time constant ratio.

The process model is taken as

$$\tilde{p}(s) = \frac{e^{-s}}{(-s+1)}. \tag{33}$$

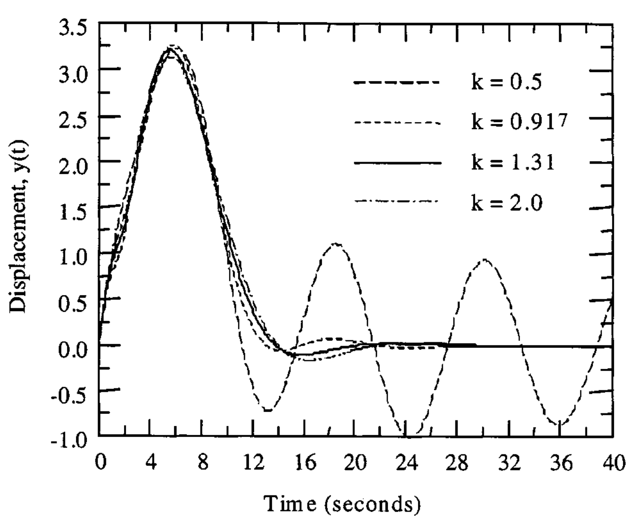

Figure 19. Response of $y(t)$ to an impulse disturbance for different spring-constant values using controller $c_\mu(s)$.

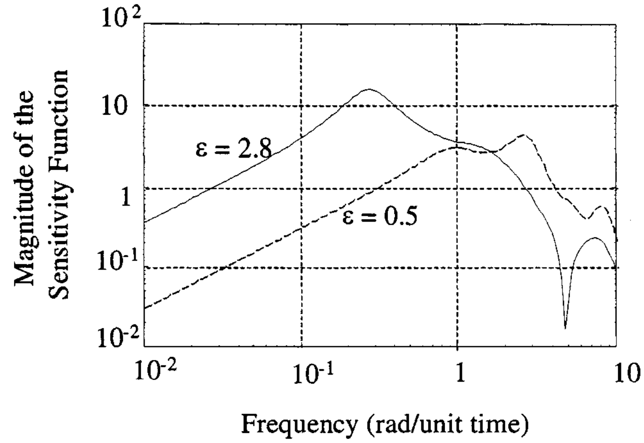

Figure 20. Sensitivity functions for Example 7 with $\epsilon = 2.8$ and 0.5.

Selecting a filter time constant, ϵ, of 0.5 gives

$$qq_d(s, 0.5) = \frac{(-s+1)(5.116s+1)}{(0.5s+1)^2}. \tag{34}$$

Substituting Eqs. 32, 33, and 34 into Eq. 16 and computing the sensitivity function yields the curve shown in Figure 20.

Figure 20 indicates that the sensitivity function for $\epsilon = 0.5$ is finite over all positive frequencies. To conclude from this that the control system is stable over all uncertain processes, however, we must also see if the control system is stable for any single process gain, K, in the range 0.9 to 1.1, but not equal to 1.0. A Nyquist analysis of the controller, $c(s)$, given by Eq. 14 with $\tilde{p}(s)$ and $qq_d(s)$ given by Eqs. 33 and 34 shows that the controller has six right half-plane poles. Further, a Nyquist analysis of the sensitivity function given by Eq. 16, using the same controller, shows that the disturbance response is unstable for a process gain of 1.1. Thus, from the Robust Stability theorem, we can conclude that the control system is unstable for all process gains in the uncertainty range. However, simply because the control system of Figure 13 is unstable for a filter time constant of 0.5 does not mean that the transfer function given by Eq. 16 with $\tilde{p}(s)$ and $qq_d(s)$ given by Eqs. 33 and 34 cannot be realized. As shown by Figure 20, the transfer function itself is stable. Use of an internally stable model-based control system such as that proposed by Berber and Brosilow (1997, 1999) allows the transfer function to be realized in a practical control system. The step disturbance and setpoint responses are shown in Figures 21 and 22.

In order to obtain a stable control system of the form shown in Figure 13, it is necessary to increase the filter time constant, ϵ, from 0.5 to 2.8. In this case, the two-degrees-of-freedom IMC controller becomes

$$qq_d(s,2.8) = \frac{(-s+1)(38.25s+1)}{(2.8s+1)^2}. \tag{35}$$

A Nyguist analysis of the controller formed by substituting Eq. 35 into Eq. 14 shows that the denominator has one right half-plane pole, but this pole is exactly at 1, and so is can-

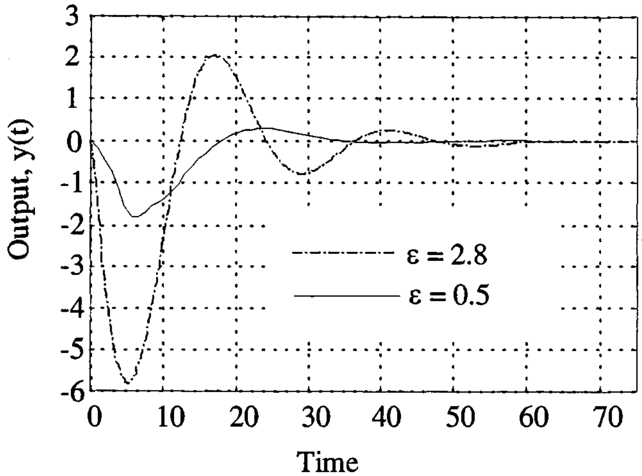

Figure 21. Output responses to a step disturbance for $\epsilon = 0.5$ and 2.8.

celed by the numerator pole at 1. This cancellation must be enforced so that the controller that is actually implemented is stable, except, of course, for the pole at the origin. Nyquist analysis of the entire control system for a process gain of 1.1 shows that the control system in stable. We can thus conclude that the control system is stable for all process gains. Step disturbance and setpoint responses for a filter time constant of 2.8 are also shown in Figures 21 and 22. Clearly, there is a significant advantage in implementing the control system with an internally stable model-based algorithm.

Conclusions

Mp tuning and synthesis is a viable alternative to other H_∞ methods, at least for single-input, single-output processes. It is relatively easy to use, and gives sharp results on the amount of overshoot to step setpoint changes, as well as on the range of response times for a very large number of tuning and design problems.

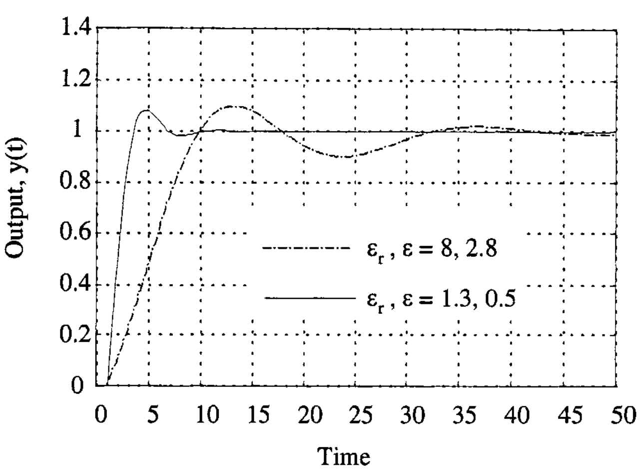

Figure 22. Responses to a step setpoint change for ϵ_r, $\epsilon = 1.3$, 0.5 and ϵ_r, $\epsilon = 8$, 2.8.

Using Mp tuning and synthesis we are able to demonstrate that process uncertainty can strongly affect the choice of IMC controller. For example, our analysis of an overdamped process of relative order two with a relatively modest amount of uncertainty demonstrates the validity of the traditional engineering approach of fitting such a process with a first-order-plus-dead-time model. Designing the controller based on that model yields nearly optimal control-system performance. Based on this example, and others not described in this article, we believe that the same conclusion holds for any overdamped process with relative order greater than two.

For uncertain unstable processes, not only the choice of model but also the choice of control structure can have a profound effect on control-system performance.

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Appendix A: IMCTUNE

Functionality

Input Data

(1) The Process

(a) Entered as either one, or the sum of two transfer functions, each transfer function is the product of polynomials cascaded with a dead time. The polynomials are supplied by the user as a matrix, each row of which contains the coefficients of one polynomial. Each coefficient can be either a constant or an uncertain parameter. The same uncertain parameter may appear at several times.

(b) Entered as a string in terms of the Laplace variable s, and the IMC filter time constant e and uncertain parameters, $x(i)$, $i = 1, 2, \ldots$.

(2) The Model

(a) Same as 1a, except that all polynomial coefficients are constants.

(b) Entered as a string in terms of the Laplace variable s and the IMC filter time constant e.

(3) The IMC Controller

(a) Entered as the portion of the process model that is inverted to form the controller. Again the polynomial coefficients are entered as elements of a matrix. In addition the user enters the order of the controller filter and time constant.

(b) The controller is entered directly as a string in terms of the Laplace variable s and the IMC filter time constant e.

(4) Process Uncertainty Bounds. The user supplies vectors of the upper and lower bounds of the uncertain parameters.

(5) Control Effort Bounds. The user supplies the upper and lower bounds on the control effort.

(6) Default Values. The users may temporarily change the following default values:

- Noise amplification factor (20) (used to set a lower bound on the IMC filter time constant)
- Frequency range (10^{-1} to 10) (used to plot frequency responses)
- Number of frequency points per decade (10)
- Accuracy of Mp tuning calculation (± 0.005)
- Number taken as infinity for Mp calculation (100,000)

Computations.

(1) *One-Degree-of-Freedom Tuning.* Computes the IMC filter time constant for a user-specified Mp and maximum peak height. The user is prompted to change current default values if desired. Input data are 1a or 1b, 2a or 2b, 3a or 3b, and 4.

(2) *Two-Degrees-of-Freedom Tuning.* Computes the IMC filter time constants for the inner loop first, and then the outer loop. The user is prompted to change current default values if desired. Input data are 1a or 1b, 2a or 2b, 3a or 3b, and 4, plus process and disturbance lag polynomial vectors.

(3) *One-Degree of Freedom PID Controllers.* Given a filter time constant, converts IMC controller and model into a simple PID controller, or a PID controller cascaded with a first-order lag, or a PID controller cascaded with a second-order lag. Plots all output and control effort responses. Input must be 2a and 3a. No other data are needed.

(4) *Two-Degrees-of-Freedom PID Controller.* Given the feed-forward and feedback IMC controllers and the process model, converts these into a setpoint filter and a simple PID controller, or a PID controller cascaded with a first-order lag, or a PID controller cascaded with a first second lag. Plots setpoint and disturbance responses, and/or control effort responses. Input must be 2a and 3a plus disturbance lag polynomial vector.

(5) *Model State Feedback Coefficients* (see Coulibably et al., 1995). Input must be 2a and 3a. No other data are needed.

(6) *Closed-Loop Frequency Responses for One- and Two-Degrees-of-Freedom Control Systems.* Computes upper and/or lower bounds for the complementary sensitivity function, the sensitivity function, the partial sensitivity function, and the integrated sensitivity function. Also plots the closed-loop response for any of the foregoing for any particular plant in the uncertainty set. Input data are 1a or 1b, 2a or 2b, 3a or 3b, 4, and, if desired, a vector specifying the uncertain parameters.

(7) *The Time Responses for Any Process in the Uncertainty Set to Step Setpoint Changes or Disturbances for Single- and Two-Degrees-of-Freedom IMC, PID, and MSF Control Systems.* Input data are 1a, 2a, 3a, 5, and a vector specifying the uncertain parameters.

(8) *Both Inner- and Outer-Loop Tuning for a Cascade Control System, Including All of the Above Functionality.* Input data are the same as previously for both inner- and outer-loop processes.

The algorithm used to compute the IMC controller filter time constant to achieve a specified Mp

Input Data.

(1) A nominal process model (either parametric or frequency-response gain and phase over the frequency range)

(2) An uncertainty description of the process parameters

(3) An initial value for the filter time constant

Default: that value of filter time constant that satisfies specification (5)

(4) An Mp specification and Tolerance:

Default: Mp = 1.05; Tolerance = ± 0.005

(5) The maximum allowable high-frequency controller noise amplification (our criterion for noise amplification follows that used for the derivative action of PID controllers, where the derivative term is approximated by a lead whose maximum high-frequency gain is ordinarily limited to between 10 and 20 so as to limit high-frequency noise amplification) (that is, $|q(\infty, \epsilon)/q0, \epsilon)|$) Default: 20

(6) An upper and lower bound for the frequency range, a number of points, and whether the distribution of points is linear or logarithmic

(a) Default for parametric nominal model:

- Low frequency: reciprocal of ten times the largest time constant or dead time
- High frequency: one thousand times the low frequency
- Number of points and scale for plotting: 30, logarithmic

(b) Default for frequency response nominal mode:

- Low frequency: one-tenth the break frequency
- High frequency: hundred times the break frequency
- Number of points and scale for plotting: 30, logarithmic

Computations.

Step 1: Initialization. Find a filter time constant ϵ such that the IMC controller does not amplify noise by more than the specified maximum noise amplification factor. This step corresponds to IMC controller design for a perfect model. The starting value of frequency is $\omega = 1/\epsilon$, and the starting values of the uncertain parameters are their upper limits.

Step 2: Maximization of $|\eta|$. Find the global maximum of $|\eta|$ over the set of uncertain parameters or the process gain and phase bounds and over all frequencies greater than zero. (IMCTUNE currently uses the constrained optimization algorithm in the MATLAB optimization toolbox. This algorithm requires bounds on the frequency, which are selected as plus and minus two decades around the current starting value of frequency of $1/\epsilon$.) If the maximum so obtained is equal to or smaller than (1 + Tolerance) times the specified Mp, then the algorithm goes to Step 4. Otherwise set the values of the frequency and the parameters at which the maximum was found to ω^* and α^* and continue.

Step 3: Finding a New Iterate of ϵ. Find a new filter time constant ϵ such that for ω^* and α^* the maximum magnitude of η reaches a specified *Mp*. The interval $[\epsilon_L; \epsilon_U]$ is found using a bracketing procedure such that $|\eta(\omega^*, \alpha^*, \epsilon_L)| > Mp$ and $|\eta(\omega^*, \alpha^*, \epsilon_U)| < Mp$. A line search is then performed using interval halving method. Return to Step 2.

Step 4: Global Solution Test. Verify that the current maximum is the global optimum. The "global solution test" algorithm is based on cross-examination of the solution. Choose values of parameters that maximize $|\eta|$ at a high frequency as a starting point. If the computed maximum is not larger than the one originally found, compute the upper bound of frequency responses over the entire frequency range using the user-specified number of points per decade. If there are any local maxima within the frequency range, explore each local maximum to find the actual peak of that maximum. (In IMCTUNE the constrained optimization is rerun at each local maximum.) If the new optimum is larger than (1 + Tolerance) times the specified Mp, use the new ω^* and α^* as starting parameters and go to Step 2. If the new optimum is smaller than (1 − Tolerance) times the specified Mp, use the new ω^* and α^* as starting parameters and go to Step 3. Otherwise, the algorithm ends, and the Mp found is considered to be the global optimum and corresponding filter time constant and an IMC controller satisfy given requirements.

Output Data.

(1) The filter time constant that achieves the specified Mp ± Tolerance and the value of Mp achieved.

(2) A curve of the upper bound of the magnitude of the complementary sensitivity function for the component value of the filter time constant.

(3) Tables listing which processes give the upper bound at each frequency.

(4) The lower bound of frequency responses for a filter time constant ϵ and time responses are also available after additional computations.

Appendix B

In order to be able to achieve any Mp specification greater than one, it is sufficient that the maximum peak of the complementary sensitivity function approach 1 as the controller filter time constant approaches infinity. As we show below, the maximum peak approaches 1 only if the model gain is chosen to be in the upper portion of the process gain uncertainty region, the exact region depending on the relative order of the finite-dimensional part of the model.

For large values of the filter time constant, the maximum peak of the complementary sensitivity function, if any, will occur at low frequencies. For low enough frequencies, the frequency response of all processes in the set Π can be approximated by their steady-state gains. That is,

$$p(i\omega) \cong p(0) \qquad \text{for} \quad \omega < \delta. \tag{B1}$$

Similarly,

$$\tilde{p}(i\omega) \cong \tilde{p}(0) \qquad \text{for} \quad \omega < \delta. \tag{B2}$$

Therefore for the controller filter time constant large enough so that

$$\epsilon\delta \gg 1, \tag{B3}$$

the terms pq can be approximated as

$$pq(s,\epsilon) \cong \frac{p(0)/\tilde{p}(0)}{(\epsilon s+1)^r} \qquad s = i\omega, \qquad \omega < \delta \tag{B4}$$

$$\tilde{p}q(s,\epsilon) \cong \frac{1}{(\epsilon s+1)^r} \qquad s = i\omega, \qquad \omega < \delta. \tag{B5}$$

Substituting Eqs. B4 and B5 into the expression for the complementary sensitivity function given by Example 2 gives

$$\eta(s) \cong \frac{\dfrac{p(0)}{\tilde{p}(0)(\epsilon s+1)^r}}{1+\left(\dfrac{p(0)}{\tilde{p}(0)}-1\right)\dfrac{1}{(\epsilon s+1)^r}} \qquad s = i\omega, \qquad \omega < \delta$$

$$\cong \frac{1}{\dfrac{\tilde{p}(0)}{p(0)}\left[(\epsilon s+1)^r - 1\right]+1}. \tag{B6}$$

The behavior of the complementary sensitivity function given by Eq. B6 depends only on the ratio $p(0)/\tilde{p}(0)$ over the frequency range $0 \le \omega < \delta$. The filter time constant, ϵ, in Eq. B6 is only a scaling factor and has no effect on the maximum magnitude ratio. For values of frequency near δ, the magnitude of η will be much less than one because $\epsilon\delta$ is much greater than one by assumption given in Eq. B3. However, for some frequencies less than δ the magnitude of η can exceed one, depending on the value of the ratio $p(0)/\tilde{p}(0)$ and the relative order r. Since we wish to be able to solve the problem given in the section on Mp tuning for any specification of Mp greater than 1, it is necessary to place an upper bound on the ratio of $p(0)/\tilde{p}(0)$ for relative orders greater than one, as shown in Table 2.

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